NAME: ______________________ **SCIPER:** ____________

This exam is open book and open notes, but no computers are allowed.

Please answer all questions. Values of each question are given below.

Problem:	1	2	3	4	5	6	Total
Value:	20	20	15	15	15	15	100
Grade:							

Problem 1.

When there are multiple choices in the following, select all statements that are true.

a) Is the union of a finite set of ellipses convex?

- ◯ Yes
- ◯ No
- ◯ Not enough information

b) Is the intersection of an ellipse and a polytope convex?

- ◯ Yes
- ◯ No
- ◯ Not enough information

c) If f is a convex function and x is a point such that $\nabla f(x) = 0$, then necessarily x is a

- ◯ local minimum of f
- ◯ global minimum of f
- ◯ global minimum of f if $\nabla^2 f(x) \succ 0$
- ◯ global minimum of f if $\nabla^2 f(x) \prec 0$

d) Let the set S be $\{(1,1), (-1,0), (0,0), (-1,-1)\}$.

- ◯ The point $(-1/3, 0)$ is contained in the convex hull of S
- ◯ The point $(0, -1/3)$ is contained in the convex hull of S

e) Consider the nominal system $x^+ = f(x)$ and let S be a non-empty set such that $S \subseteq \text{pre}(S)$. Under which of the following conditions is S an invariant set of the uncertain system $y^+ = f(y) + w$ where w is restricted to lie in W? (Note that the pre-set operator below refers to the nominal system)

- ◯ $S \subseteq \text{pre}(S \ominus W)$
- ◯ $S \oplus W \subseteq \text{pre}(S)$
- ◯ $S \subseteq \text{pre}(S)$
- ◯ $S \subseteq \text{pre}(S) \oplus W$

(Recall that $\ominus$ is the Pontryagin difference, and $\oplus$ is the Minkowski sum.)

f) The function $\cosh x = (e^x + e^{-x})/2$ is

- ◯ Convex
- ◯ Concave
- ◯ Neither convex nor concave
- ◯ Affine

g) On the domain $y > 0$, the function $-x^2/y$ is

- ◯ Convex
- ◯ Concave
- ◯ Neither convex nor concave
- ◯ Affine

h) The function $\max\{x + 4,\ 2x\}$ is

- ◯ Convex
- ◯ Concave
- ◯ Neither convex nor concave
- ◯ Affine

i) Consider the system $x^+ = -0.5x$. Which of the following sets are invariant?

- ◯ $\{x \mid x^4 \leq 5\}$
- ◯ $\{x \mid x^3 \leq 5\}$
- ◯ $\{x \mid -1 \leq x \leq 2\}$
- ◯ $\{x \mid -1/2 \leq x \leq 2\}$

j) Consider the system

$$\begin{pmatrix} x_1^+ \\ x_2^+ \end{pmatrix} = \begin{pmatrix} 0.5x_1 \\ 1.5x_2 \end{pmatrix}$$

Which of the following sets is invariant?

- ◯ $\{x \mid x_2 = 0,\ x_1 \leq 10\}$
- ◯ $\{x \mid x_1 = 0,\ x_2 \leq 10\}$
- ◯ $\{x \mid x_2 = x_1\}$

k) Consider the following predictive control problem, which defines the receding horizon control law $\pi(x)$

$$\begin{aligned} \min \ & \sum_{i=0}^{N} x_i^T Q x_i + u_i^T R u_i \\ \text{s.t. } & x_{i+1} = Ax_i + Bu_i \\ & x_0 = x \end{aligned}$$

Which of the following statements is true:

- ◯ The control law $\pi(x)$ is quadratic
- ◯ The control law $\pi(x)$ is linear
- ◯ If the closed-loop system is stable for $N = 5$, then it will be stable for $N = 6$
- ◯ The closed-loop system will be stable if $x^+ = Ax$ is unstable and $N \geq \text{rank}(A)$

l) Consider the following predictive control problem,

$$\begin{aligned} \min \ & \sum_{i=0}^{N} x_i^T Q x_i + u_i^T R u_i \\ \text{s.t. } & x_{i+1} = Ax_i + Bu_i \\ & x_i \in X,\ u_i \in U \\ & x_0 = x \end{aligned}$$

Which of the following statements is true:

- ◯ The feasible set of the above optimization problem is larger for $N = 5$, than for $N = 4$
- ◯ If the closed-loop system is stable for $N = 5$, then it will be stable for $N = 6$
- ◯ If the MPC problem is recursively feasible for $N = 5$, then is will be for $N = 6$
- ◯ None of the above

m) What is the proximal operator $\text{prox}_{f,\rho}(v)$ of the function $f(x) = \|Ax - b\|_2^2$?

- ◯ v
- ◯ $Av - b$
- ◯ $(A^T A + \rho I)^{-1}(A^T b + v)$
- ◯ $(2A^T A + \rho I)^{-1}(2A^T b + \rho v)$

n) What is the proximal operator $\text{prox}_{f,\rho}(v)$ of the function

$$f(x) = \begin{cases} 0 & x = 0 \\ 0 & x = 1 \\ \infty & \text{otherwise} \end{cases}$$

- ◯ ρv
- ◯ v
- ◯ $\begin{cases} 0 & \text{if } v \leq 0.5 \\ 1 & \text{if } v > 0.5 \end{cases}$
- ◯ ∞

o) Let $f(x)$ be the indicator function for the convex set C. What is $\text{prox}_{f,\rho}(v)$ for a point $v \in C$?

- ◯ ρv
- ◯ v
- ◯ ρC
- ◯ ∞

p) Consider the standard (right) and soft-constrained (left) MPC problem formulations:

$$\begin{aligned} J^\star_{soft}(x) = \min_u \; & \sum_{i=0}^{N-1} l(x_i,u_i) + V_N(x_N) + \rho \sum_{i=0}^{N-1} \epsilon_i^T \epsilon_i \\ \text{s.t.} \; & x_{i+1} = Ax_i + Bu_i \\ & Gx_i \le g + \epsilon_i \\ & Hu_i \le h \\ & \epsilon_i \ge 0 \end{aligned} \qquad \Bigg| \qquad \begin{aligned} J^\star(x) = \min_u \; & \sum_{i=0}^{N-1} l(x_i,u_i) + V_N(x_N) \\ \text{s.t.} \; & x_{i+1} = Ax_i + Bu_i \\ & Gx_i \le g \\ & Hu_i \le h \end{aligned}$$

where the standard problem has been designed with appropriate terminal weights and constraints so that the resulting problem is recursively feasible and $J^\star(x)$ is a Lyapunov function. Let Z be the set of states for which the standard problem is feasible, and $\pi_{soft}(x)$ the control law resulting from solving the soft-constrained problem. Which of the following conditions will be satisfied:

- ◯ $J^\star_{soft}(x) \le J^\star(x)$ for all $x \in Z$
- ◯ $J^\star_{soft}(Ax + B\pi_{soft}(x)) \le J^\star_{soft}(x)$ for all $x \in Z$
- ◯ $J^\star_{soft}(Ax + B\pi_{soft}(x)) \le J^\star_{soft}(x)$ for all $x \notin Z$

q) Consider the MPC control law for the linear system $x^+ = Ax + Bu$

$$\begin{aligned} \min \; & \sum_{i=0}^{N} l(x_i,u_i) + x_N^T P x_N \\ \text{s.t.} \; & x_{i+1} = Ax_i + Bu_i \\ & x_i \in X, \; u_i \in U \\ & x_N \in X_f \\ & x_0 = x \end{aligned}$$

Where K is a matrix such that the set $X_f \subset X$ is invariant for $x^+ = (A + BK)x$, $KX_f \subset U$, and the function $l(x,u)$ is positive definite. Which of the following additional conditions will ensure asymptotic stability of the closed-loop system?

- ◯ $(A + BK)^T P(A + BK) - P \preceq 0$
- ◯ $x^T[(A + BK)^T P(A + BK) - P]x \le -l(x, Kx)$ for all $x \in X_f$
- ◯ X_f is a control invariant set

r) Which of the following statements implies that $S = \left\{x \;\middle|\; x^T Px \le 1\right\}$, $P \succeq 0$ is an invariant set for the system $x^+ = Ax$?

- ◯ $A^T PA \succeq P$
- ◯ $A^T PA \preceq P$
- ◯ $A^T PA \succeq 0$
- ◯ $A^T PA \preceq 0$

Problem 2.

/20

Consider the following quadratically constrained quadratic program:

$$\begin{aligned} &\min \frac{1}{2}x^T Q x + c^T x \\ &\text{s.t. } x^T x \leq \alpha \end{aligned} \tag{1}$$

where $Q \succ 0$ is a positive definite matrix.

a) Barrier method

i) Consider the barrier function $\phi(x) = -\sum_{i=1}^{m} \log(-g_i(x))$, where $g_i(x) \leq 0$ are the constraints of the problem. Compute the function $\phi(x)$, its gradient and its hessian for the optimization problem given above.

ii) Compute the Newton direction for solving the centering step of the barrier interior-point method for the above problem.

iii) Let $Q = I$, $c = \begin{pmatrix} 1 & 2 \end{pmatrix}^T$ and $\alpha = 3$. (1) Compute the Newton direction at the point $x = \begin{pmatrix} 1 & 1 \end{pmatrix}^T$ for a value of the barrier parameter $\kappa = 1$ and (2) demonstrate that the result is a decent direction.

b) Alternating Direction Method of Multipliers

$$\begin{aligned} &\min\ f(x) + g(y) \\ &\text{s.t. } Ax + By = b \end{aligned} \tag{2}$$

i) Give functions f, g and matrices A, B and b so that problem (2) is equivalent to (1).
Hint: You may want to use an indicator function.

ii) Give the three steps of the ADMM algorithm for the functions and data you gave in part i)

$$\begin{aligned} x^{k+1} &= \text{argmin}_x\ \ f(x) + \frac{\rho}{2}\|Ax + By^k - b + \mu^k\|^2 \\ y^{k+1} &= \text{argmin}_y\ \ g(y) + \frac{\rho}{2}\|Ax^{k+1} + By - b + \mu^k\|^2 \\ \mu^{k+1} &= \mu^k + Ax^{k+1} + By^{k+1} - b \end{aligned}$$

Problem 3.

/15

Consider the system $x^+ = Ax + Bu$ with the state constraint $x \in X$ and input constraint $u \in U$.

Let $C \subseteq X$ be a control invariant set for this system and consider the following MPC controller.

$$
\begin{aligned}
\min \quad & \sum_{i=0}^{N-1} x_i^T Q x_i + u_i^T R u_i \\
\text{s.t.} \quad & x_1 \in C \\
& u_i \in U \quad i \in \{0, \dots, N-1\} \\
& x_i \in 2 \cdot X \quad i \in \{1, \dots, N\} \\
& x_{i+1} = Ax_i + Bu_i
\end{aligned}
$$

Is the resulting closed-loop system recursively feasible?

◯ Yes ◯ No

If yes, then prove it. If no, then provide a counter-example.

Problem 4.

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Consider the linear system

$$x^+ = \begin{bmatrix} 0.5 & 0 \\ 4 & 0.8 \end{bmatrix} x + \begin{bmatrix} 0.3 & 0.2 \\ -0.6 & 0.9 \end{bmatrix} u$$

with constraints on the input $\|u\|_\infty \leq 1$.

a) What is the maximum control invariant set for this system? Justify your answer.

b) Consider the following standard MPC optimization problem, and let $\pi(x)$ be the resulting receding-horizon control law.

$$\begin{aligned} \min \ & \sum_{i=0}^{N-1} x_i^T Q x_i + u_i^T R u_i + x_N^T Q_f x_N \\ \text{s.t. } & x_{i+1} = A x_i + B u_i \\ & u_i \in U \quad i \in \{0, \dots, N-1\} \\ & x_N \in X_f \\ & x_0 = x \end{aligned}$$

Describe how to choose a terminal control law, K_f, terminal weight Q_f and terminal set X_f so that the closed-loop system $x^+ = Ax + B\pi(x)$ has a maximal invariant set equal to that given in Part a) and is asymptotically stable.

Problem 5.

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Consider the uncertain system $x^+ = \frac{1}{2}x + w$ under the state constraint $-10 \leq x \leq 10$ and subject to a disturbance bounded to lie in the set $|w| \leq 1$.

a) Give an algorithm to compute the minimum robust invariant set

b) Compute the minimum robust invariant set
Hint: $[a, b] \oplus [c, d] = [a + c, b + d]$

c) Give an algorithm to compute the maximum robust invariant set

d) Compute the maximum robust invariant set

Problem 6.

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Consider the following linear complementarity problem:

$$w - \begin{bmatrix} 5 & 7 \\ 7 & 10 \end{bmatrix} z = q \quad w^T z = 0 \quad w, z \geq 0$$

a) What is the solution to this LCP for $q = \bar{q} = \begin{bmatrix} -1 & 2 \end{bmatrix}^T$?

b) Find a matrix T and a vector t such that $\begin{bmatrix} w \\ z \end{bmatrix} = Tq + t$ is the solution to the LCP in a neighbourhood of $\bar{q}$

c) Give the neighbourhood P in which the affine function you found in the last part is the solution to the LCP.